Valentyn Khokhlov, CFA, PhD in Technical Sciences
International Marketing Manager, Global Spirits, Ukraine

CONDITIONAL VALUE-AT-RISK FOR ELLIPTICAL DISTRIBUTIONS

Conditional Value-at-Risk (cVaR) represents a significant improvement over the Value-at-Risk (VaR) in the area of risk measurement, as it catches the risk beyond the VaR threshold. cVaR is also theoretically more solid, being a coherent risk measure, which enables building more robust risk assessment and management systems. However, in its general form cVaR calculations require sophisticated mathematical software, whereas VaR can be easily calculated using spreadsheets. In this paper closed-form cVaR formulas are derived for the most important elliptical distributions, such as normal, Student's t-distribution, Laplace and logistic distributions. These closed-form formulas not only enable practitioners to estimate cVaR in spreadsheets, but also allow developing portfolio optimization models by providing closed-form objective functions for them. The results of variance-covariance cVaR estimation were compared with the historical cVaR, and the relative estimation error is between 6% and 10%, which is comparable with VaR estimation error.

Key words: Value-at-Risk, conditional Value-at-Risk, risk assessment, elliptical distributions, Laplace, logistic distributions.

Introduction

Conditional value-at-risk (cVaR) has recently become a popular measure of risk. While less known than value-at-risk (VaR), cVaR has an important advantage of being a coherent risk measure as defined by Artzner[1]. Another serious drawback of VaR is its inability to quantify the expected losses beyond the threshold amount, i.e. VaR only allows identifying the threshold itself but says nothing about the worst-case scenarios below it. However, VaR remains the most popular risk measure and is widely covered by academics and practitioners, such as Jorion[2], Stambaugh[3], Linsmeier[4]. cVaR, on the other hand, has been mostly overlooked by practitioners because it lacks an easy-to-use straightforward formula.

Starting with Rockafellar and Uryasev[5] there were attempts of simplifying the generic cVaR formula for some distributions and deriving it in a closed form. Rockafellar and Uryasev[6] did this for the normal distribution using the error function, which unfortunately is not present in most non-mathematical software (e.g. Microsoft Excel™). Andreev et al.[7] derived closed-form cVaR formulas for many distributions, however they didn't consider a generic normal and generalized Student's t-distributions — the most relevant distributions for modeling risk. Furthermore, they used a non-standard representation for the Laplace distribution.

The purpose of this paper is to derive analytically closed-form cVaR formulas for the most popular elliptical distributions. We consider the normal distribution, generalized Student's t-distribution, the Laplace distribution and the logistic distribution. The choice is limited to these distributions, because they can be defined with two parameters (location and scale), as used in the Markowitz portfolio optimization problem. Our methodology is based on the variance-covariance definition of cVaR, so all the formulas are derived under this approach. However, we also compare the results of our cVaR estimation with cVaR calculated under the empirical (historical) approach.

---

[1] Artzner, P., Delbaen, F. Eber, J.-M., Heath, D. (1999). Coherent Measures of Risk. *Mathematical Finance, Vol. 9, No. 3*, 203–228.
[2] Jorion, P. (1996). Risk: Measuring the Risk in Value at Risk. *Financial Analysts Journal, Vol. 52, No. 6*, 47–56.
[3] Stambaugh, F. (1996). Risk and Value at Risk. *European Management Journal, Vol. 14, No. 6*, 612–621.
[4] Linsmeier, T., Pearson, N.D. (2000). Value at Risk. *Financial Analysts Journal, Vol. 56, No. 2*, 47–67.
[5] Rockafellar, R.T., Uryasev, S. (2000). Optimization of conditional value-at-risk. *Journal of Risk, Vol. 2*, 21-41.
[6] Rockafellar, R.T., Uryasev, S. (2000). Optimization of conditional value-at-risk. *Journal of Risk, Vol. 2*, 21-41.
[7] Andreev, A., Kanto, A., Malo, P. (2005). On closed-form calculation of cVaR. *Helsinki School of Economics Working Paper W-389*. <http://epub.lib.aalto.fi/pdf/wp/w389.pdf> (2016, October, 11)

Two definitions of cVaR

cVaR is defined as the conditional expectation of losses above VaR. Under the variance-covariance approach, we assume that the return is a random variable that follows a certain distribution, and the cVaR per $1 is calculated as

$$-c_\alpha = \frac{1}{\alpha}\int_{-\infty}^{-v_\alpha} x\,\mathrm{f}(x)\,dx, \tag{1}$$

where c_α is the cVaR @ α per $1 of the selected asset,

v_α is the VaR @ α per $1 of the selected asset,

x is the return on the selected asset (random variable),

$\mathrm{f}(x)$ is the probability density function (PDF) for the distribution of return.

Under the same approach, VaR is defined using the inverse cumulative distribution function (CDF), which for elliptical distributions can be expanded using the location and scale parameters and the standard CDF as follows:

$$-v_\alpha = \mathrm{F}^{-1}(\alpha) = \mu + \sigma_s\Phi^{-1}(\alpha), \tag{2}$$

where v_α is the VaR @ α per $1 of the selected asset,

$\mathrm{F}^{-1}(\alpha)$ is the standard CDF value at α,

μ, σ_s are the location and the scale parameter of the distribution of return,

$\Phi^{-1}(\alpha)$ is the inverse standard CDF value at α.

We put a minus sign in front of the left-hand side in (1) and (2) in order to have the values of VaR and cVaR positive (as follows from their definitions), whereas the right-hand sides are expressed in terms of return, which is negative in case of losses.

Under the empirical (historical) approach, cVaR is calculated as the conditional sample mean for the values that are less than VaR:

$$-c_\alpha^{hist} = \frac{1}{n}\sum_{r_i < -v_\alpha^{hist}} r_i, \tag{3}$$

where c_α^{hist} is the historical cVaR @ α per $1,

v_α^{hist} is the historical VaR @ α per $1, which is the α-th percentile,

r_i is the i-th value in the sample of historical returns ($i = 1,\ldots,N$).

n is the amount of returns that are less than $-v_\alpha^{hist}$, i.e. $n = \lceil \alpha N \rceil$.

Derivation of cVaR formulas

The derivation of closed-form cVaR formulas is based on analytical expansion of the integral in (1). Let's start with the normal distribution that has its PDF defined as follows:

$$\mathrm{f}(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}},$$

where μ is the location parameter (expected value),

σ is the scale parameter, which equals to the standard deviation.

Proposition 1. For the normal distribution the cVaR @ α can be calculated as

$$-c_\alpha = \mu - \sigma\frac{\phi\left(\Phi^{-1}(\alpha)\right)}{\alpha}, \tag{4}$$

where μ, σ are the location and the scale parameter of the distribution of return,

$\phi(x)$ is the standard PDF: $\phi(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$,

$\Phi^{-1}(\alpha)$ is the inverse standard CDF value at α.

Proof: Let's start with the derivative of the PDF:

$$\frac{d}{dx}\mathrm{f}(x)=-\frac{1}{\sigma^2}(x-\mu)\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}=-\frac{x\mathrm{f}(x)}{\sigma^2}+\frac{\mu\mathrm{f}(x)}{\sigma^2},$$

therefore, $x\mathrm{f}(x)dx=\mu\mathrm{f}(x)dx-\sigma^2 d\mathrm{f}(x)$, so we can take the indefinite integral of both sides: $\int x\mathrm{f}(x)dx=\int\mu\mathrm{f}(x)dx-\int\sigma^2 d\mathrm{f}(x)=\mu\mathrm{F}(x)-\sigma^2\mathrm{f}(x)$. Using this expression, we can now calculate the definite integral in (1):

$\int_{-\infty}^{-v_\alpha} x\mathrm{f}(x)dx=\mu\mathrm{F}(-v_\alpha)-\sigma^2\mathrm{f}(-v_\alpha)-\lim_{x\to-\infty}\mu\mathrm{F}(x)+\lim_{x\to-\infty}\sigma^2\mathrm{f}(x)$. Since for any distribution we have $\lim_{x\to-\infty}\mathrm{F}(x)=\lim_{x\to-\infty}\mathrm{f}(x)=0$, cVaR can be calculated as: $-c_\alpha=\frac{1}{\alpha}\int_{-\infty}^{-v_\alpha}x\mathrm{f}(x)dx=\frac{\mu}{\alpha}\mathrm{F}(-v_\alpha)-\frac{\sigma^2}{\alpha}\mathrm{f}(-v_\alpha)$. Substituting (2) into this expression,

$$-c_\alpha=\frac{\mu}{\alpha}\alpha-\frac{\sigma^2}{\alpha}\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{\left(\mu+\sigma\Phi^{-1}(\alpha)-\mu\right)^2}{2\sigma^2}}=\mu-\frac{\sigma}{\alpha}\frac{1}{\sqrt{2\pi}}e^{-\frac{\left(\Phi^{-1}(\alpha)\right)^2}{2}}=\mu-\sigma\frac{\phi\left(\Phi^{-1}(\alpha)\right)}{\alpha}.$$

Formula (4) can easily be implemented in spreadsheets, for example in Microsoft Excel™ $\Phi^{-1}(\alpha)$ is NORMSINV(α) and $\phi(x)$ is NORMDIST(*x*,0,1,0).

The generalized Student's t-distribution has its PDF defined as

$$\mathrm{f}(x)=\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi\nu}\sigma}\left(1+\frac{1}{\nu}\left(\frac{x-\mu}{\sigma}\right)^2\right)^{-\frac{\nu+1}{2}},$$

where Γ is the gamma function,
ν is the amount of the degrees of freedom,
μ is the location parameter (expected value),
σ is the scale parameter (note that the standard deviation is $\sigma\sqrt{\nu}/\sqrt{\nu-2}$).

Proposition 2. For the Student's t-distribution the cVaR @ α can be calculated as

$$-c_\alpha=\mu-\sigma\frac{\nu+\left(\mathrm{T}^{-1}(\alpha)\right)^2}{\nu-1}\frac{\tau\left(\mathrm{T}^{-1}(\alpha)\right)}{\alpha}, \tag{5}$$

where μ,σ are the location and the scale parameter of the distribution of return,

$\tau(x)$ is the standard PDF: $\tau(x)=\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi\nu}}\left(1+\frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}$,

$T^{-1}(\alpha)$ is the inverse standard CDF value at α.

Proof: Again, let's start with the derivative of the PDF:

$$\frac{d}{dx}f(x)=\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi\nu}\sigma}\left(-\frac{\nu+1}{2}\right)\left(1+\frac{1}{\nu}\left(\frac{x-\mu}{\sigma}\right)^2\right)^{-\frac{\nu+1}{2}-1}\frac{2}{\nu\sigma}\left(\frac{x-\mu}{\sigma}\right)=$$

$$=f(x)\left(-\frac{\nu+1}{2}\right)\left(1+\frac{(x-\mu)^2}{\nu\sigma^2}\right)^{-1}\frac{2}{\nu\sigma}\left(\frac{x-\mu}{\sigma}\right)=$$

$$=-f(x)\frac{\nu+1}{2}\frac{\nu\sigma^2}{\nu\sigma^2+(x-\mu)^2}\frac{2}{\nu\sigma^2}(x-\mu)=-f(x)\frac{\nu+1}{\nu\sigma^2+(x-\mu)^2}(x-\mu)$$

therefore, $\nu\sigma^2 d\,f(x)+(x-\mu)^2 d\,f(x)=-(\nu+1)x\,f(x)dx+\mu(\nu+1)f(x)dx$, and $x\,f(x)dx=-\frac{\nu}{\nu+1}\sigma^2 d\,f(x)-\frac{1}{\nu+1}(x-\mu)^2 d\,f(x)+\mu f(x)dx$, so we can take the indefinite integral of both sides:

$$\int x\,f(x)dx=-\frac{\nu}{\nu+1}\sigma^2\int d\,f(x)-\frac{1}{\nu+1}\int(x-\mu)^2 d\,f(x)+\mu\int f(x)dx=$$
$$=-\frac{\nu}{\nu+1}\sigma^2 f(x)-\frac{1}{\nu+1}\int(x-\mu)^2 d\,f(x)+\mu F(x). \tag{5a}$$

Now we can use integration by parts on $\int(x-\mu)^2 d\,f(x)$:

$$\int(x-\mu)^2 d\,f(x)=(x-\mu)^2 f(x)-\int f(x)2(x-\mu)dx=$$
$$=(x-\mu)^2 f(x)-2\int x\,f(x)dx+2\mu\int f(x)dx=$$
$$=(x-\mu)^2 f(x)-2\int x\,f(x)dx+2\mu F(x). \tag{5b}$$

Now we can substitute (5b) into (5a), and then group the sub-expressions:

$$\int x\,f(x)dx=-\frac{\nu}{\nu+1}\sigma^2 f(x)-\frac{(x-\mu)^2}{\nu+1}f(x)+\frac{2}{\nu+1}\int x\,f(x)dx-\frac{2\mu}{\nu+1}F(x)+\mu F(x),$$

$$\frac{\nu-1}{\nu+1}\int x\,f(x)dx=-\frac{\nu\sigma^2+(x-\mu)^2}{\nu+1}f(x)+\mu\frac{\nu-1}{\nu+1}F(x),$$

$$\int x\,f(x)dx=-\frac{\nu\sigma^2+(x-\mu)^2}{\nu-1}f(x)+\mu F(x). \tag{5c}$$

Finally, we can use (5c) to calculate the definite integral in (1):

$$\int_{-\infty}^{-v_\alpha} x\,\mathrm{f}(x)\,dx = -\frac{\nu\sigma^2+(-v_\alpha-\mu)^2}{\nu-1}\mathrm{f}(-v_\alpha)+\mu\mathrm{F}(-v_\alpha)+\lim_{x\to-\infty}\frac{\nu\sigma^2+(x-\mu)^2}{\nu-1}\mathrm{f}(x)-\lim_{x\to-\infty}\mu\mathrm{F}(x)$$

where the last term is zero, since for any distribution $\lim_{x\to-\infty}\mathrm{F}(x)=0$, and we can also expand the third term into two sub-terms:

$$\lim_{x\to-\infty}\frac{\nu\sigma^2+(x-\mu)^2}{\nu-1}\mathrm{f}(x)=\frac{\nu\sigma^2}{\nu-1}\lim_{x\to-\infty}\mathrm{f}(x)+\frac{1}{\nu-1}\lim_{x\to-\infty}(x-\mu)^2\mathrm{f}(x)$$

where $\lim_{x\to-\infty}\mathrm{f}(x)=0$, but the second sub-term represents an infinity-zero limit case:

$$\lim_{x\to-\infty}(x-\mu)^2\mathrm{f}(x)=\lim_{x\to-\infty}(x-\mu)^2\frac{\gamma}{\left(1+\frac{1}{\nu}\left(\frac{x-\mu}{\sigma}\right)^2\right)^{\frac{\nu+1}{2}}}\text{, where }\gamma=\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi\nu}\sigma}.$$

For this sub-term we can use L'Hôpital's rule by taking derivatives from the denominator and the numerator separately:

$$\lim_{x\to-\infty}(x-\mu)^2\mathrm{f}(x)=\lim_{x\to-\infty}\frac{2\gamma(x-\mu)}{\frac{\nu+1}{2}\left(1+\frac{1}{\nu}\left(\frac{x-\mu}{\sigma}\right)^2\right)^{\frac{\nu-1}{2}}\frac{2(x-\mu)}{\nu\sigma^2}}=$$

$$=\lim_{x\to-\infty}\frac{2\nu\sigma^2}{\nu+1}\gamma\left(1+\frac{1}{\nu}\left(\frac{x-\mu}{\sigma}\right)^2\right)^{-\frac{\nu-1}{2}}=\frac{2\nu\sigma^2}{\nu+1}\lim_{x\to-\infty}\mathrm{f}(x)=0.$$

As we have shown, all limits are zero, so finally we can express cVaR as

$$-c_\alpha=\frac{\mu}{\alpha}\mathrm{F}(-v_\alpha)-\frac{\nu\sigma^2+(v_\alpha-\mu)^2}{\alpha(\nu-1)}\mathrm{f}(-v_\alpha), \tag{5d}$$

and then we can substitute (2) into (5d):

$$-c_\alpha=\mu-\frac{\nu\sigma^2+\left(\mu+\sigma\mathrm{T}^{-1}(\alpha)-\mu\right)^2}{\alpha(\nu-1)}\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi\nu}\sigma}\left(1+\frac{1}{\nu}\left(\frac{\mu+\sigma\mathrm{T}^{-1}(\alpha)-\mu}{\sigma}\right)^2\right)^{-\frac{\nu+1}{2}}=$$

$$=\mu-\sigma^2\frac{\nu+\left(\mathrm{T}^{-1}(\alpha)\right)^2}{\alpha(\nu-1)}\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi\nu}\sigma}\left(1+\frac{\left(\mathrm{T}^{-1}(\alpha)\right)^2}{\nu}\right)^{-\frac{\nu+1}{2}}=$$

$$=\mu-\sigma\frac{\nu+\left(\mathrm{T}^{-1}(\alpha)\right)^2}{\nu-1}\frac{\tau\left(\mathrm{T}^{-1}(\alpha)\right)}{\alpha}.$$

To use formula (5) in Microsoft Excel™ $\mathrm{T}^{-1}(\alpha)$ can be written as TINV(2*α,ν) and $\tau(x)$ as T.DIST(*x*,ν,0), which is available in Excel 2010.

The Laplace distribution's PDF and CDF can be expressed as follows:

$$\mathrm{f}(x)=\frac{1}{2b}e^{\frac{x-\mu}{b}},\mathrm{F}(x)=\frac{1}{2}e^{\frac{x-\mu}{b}},\mathrm{F}^{-1}(\alpha)=\mu+b\ln(2\alpha),$$

where μ is the location parameter (expected value),

b is the scale parameter, where the standard deviation $\sigma=b\sqrt{2}$.

Proposition 3. For the Laplace distribution the cVaR @ α can be calculated as

$$-c_\alpha=-v_\alpha-b=\mu-b(1-\ln 2\alpha), \tag{6}$$

where μ, b are the location and the scale parameter of the distribution of return,

v_α is the VaR @ α per \$1 of the asset's value.

Proof: Since, by definition, $d\,\mathrm{F}(x)=\mathrm{f}(x)dx$, we can use integration by parts:

$$\int x\mathrm{f}(x)dx=\int xd\,\mathrm{F}(x)=x\mathrm{F}(x)-\int\mathrm{F}(x)dx=x\frac{1}{2}e^{\frac{x-\mu}{b}}-\int\frac{1}{2}e^{\frac{x-\mu}{b}}dx=\frac{x-b}{2}e^{\frac{x-\mu}{b}},$$

so we can now calculate the definite integral in (1):

$$\int_{-\infty}^{-v_\alpha}x\mathrm{f}(x)dx=-v_\alpha\frac{1}{2}e^{\frac{-v_\alpha-\mu}{b}}-b\frac{1}{2}e^{\frac{-v_\alpha-\mu}{b}}-\lim_{x\to-\infty}x\frac{1}{2}e^{\frac{x-\mu}{b}}+\lim_{x\to-\infty}b\frac{1}{2}e^{\frac{x-\mu}{b}}$$, where the last term is zero, since $\lim_{x\to-\infty}b\frac{1}{2}e^{\frac{x-\mu}{b}}=b\lim_{x\to-\infty}\mathrm{F}(x)=0$. However, the third term represents the infinity-zero limit, and we can use L'Hôpital's rule for it:

$$\lim_{x\to-\infty}x\frac{1}{2}e^{\frac{x-\mu}{b}}=\lim_{x\to-\infty}\frac{x}{2e^{-\frac{x-\mu}{b}}}=\frac{1}{2}\lim_{x\to-\infty}\frac{1}{-\frac{2}{b}e^{-\frac{x-\mu}{b}}}=-b\lim_{x\to-\infty}\frac{1}{2}e^{\frac{x-\mu}{b}}=-b\lim_{x\to-\infty}\mathrm{F}(x)=0.$$

Thus, cVaR for the Laplace distribution can be expressed as

$$-c_\alpha=\frac{1}{\alpha}\int_{-\infty}^{-v_\alpha}x\mathrm{f}(x)dx=\frac{-v_\alpha-b}{2\alpha}e^{\frac{-v_\alpha-\mu}{b}}=\frac{-v_\alpha-b}{\alpha}\mathrm{F}(-v_\alpha). \tag{6a}$$

Substituting (2) into (6a), we can simplify it further: $-c_\alpha=-v_\alpha-b=\mu-b(1-\ln 2\alpha)$.

The logistic distribution's PDF and CDF can be expressed as follows:

$$\mathrm{f}(x)=\frac{1}{s}e^{-\frac{x-\mu}{s}}\left(1+e^{-\frac{x-\mu}{s}}\right)^{-2};\mathrm{F}(x)=\left(1+e^{-\frac{x-\mu}{s}}\right)^{-1};\mathrm{F}^{-1}(\alpha)=\mu+s\ln\left(\frac{\alpha}{1-\alpha}\right),$$

where μ is the location parameter (expected value),

s is the scale parameter, where the standard deviation $\sigma=\pi s/\sqrt{3}$.

Proposition 4. For the logistic distribution the cVaR @ α can be calculated as

$$-c_\alpha=\mu-s\ln\frac{(1-\alpha)^{1-1/\alpha}}{\alpha}, \tag{7}$$

where μ, s are the location and the scale parameter of the distribution of return.

Proof: Since, by definition, $d\,\mathrm{F}(x)=\mathrm{f}(x)dx$, we can use integration by parts:

$$\int x\,\mathrm{f}(x)dx = x\,\mathrm{F}(x) - \int \frac{dx}{1+e^{-\frac{x-\mu}{s}}}.$$

$\frac{d}{dx} s\ln\left(e^{\frac{x-\mu}{s}}+1\right) = s\frac{1}{e^{\frac{x-\mu}{s}}+1}\frac{1}{s}e^{\frac{x-\mu}{s}} = \frac{1}{1+e^{-\frac{x-\mu}{s}}}$, so $\int \frac{dx}{1+e^{-\frac{x-\mu}{s}}} = s\ln\left(e^{\frac{x-\mu}{s}}+1\right)$, and we now can calculate the definite integral in (1):

$\int_{-\infty}^{-v_\alpha} x\,\mathrm{f}(x)dx = -v_\alpha\,\mathrm{F}(-v_\alpha) - s\ln\left(e^{\frac{-v_\alpha-\mu}{s}}+1\right) - \lim_{x\to-\infty} x\,\mathrm{F}(x) + \lim_{x\to-\infty} s\ln\left(e^{\frac{x-\mu}{s}}+1\right)$, where

$\lim_{x\to-\infty} s\ln\left(e^{\frac{x-\mu}{s}}+1\right) = s\ln\left(\lim_{x\to-\infty} e^{\frac{x-\mu}{s}}+1\right) = s\ln(0+1) = 0$. However, the third term represents the infinity-zero limit, so let's use L'Hôpital's rule to approach it: $\lim_{x\to-\infty} x\,\mathrm{F}(x) = \lim_{x\to-\infty} \frac{x}{1+e^{-\frac{x-\mu}{s}}} = \lim_{x\to-\infty} \frac{1}{-\frac{1}{s}e^{-\frac{x-\mu}{s}}} = \lim_{x\to-\infty} -se^{\frac{x-\mu}{s}} = 0$.

Thus, cVaR for the logistic distribution can be expressed as

$$-c_\alpha = -\frac{v_\alpha}{\alpha}\mathrm{F}(-v_\alpha) - \frac{1}{\alpha}s\ln\left(e^{\frac{-v_\alpha-\mu}{s}}+1\right). \quad (7a)$$

Substituting (2) into (7a),

$$-c_\alpha = \mu + s\ln\left(\frac{\alpha}{1-\alpha}\right) - \frac{1}{\alpha}s\ln\left(e^{\frac{\mu+s\ln\left(\frac{\alpha}{1-\alpha}\right)-\mu}{s}}+1\right) = \mu + s\ln\left(\frac{\alpha}{1-\alpha}\right) - \frac{s}{\alpha}\ln\left(\frac{1}{1-\alpha}\right) =$$

$$= \mu - s\ln\left(\frac{1-\alpha}{\alpha}\right) - s\ln\left(\frac{1}{1-\alpha}\right)^{1/\alpha} = \mu - s\ln\frac{(1-\alpha)^{1-1/\alpha}}{\alpha}. \square$$

cVaR properties and its usage for portfolio optimization

Formulas (4)–(7) can be used to investigate properties of variance-covariance cVaR. All these formulas can be written in a generic form

$$-c_\alpha = \mu - \chi_\alpha\sigma, \quad (8)$$

where μ, σ are the expected return and its standard deviation, and χ_α is a distribution-specific value that does not depend on μ or σ.

Let's start our investigation with the Standard & Poor's (S&P) 500 index that corresponds to the value of a diversified portfolio composed of the 500 largest U.S. companies stocks. Using a sample of rolling 12-month returns from January 1951 to December 2015 (780 observations), we can calculate the average sample return of 8.8% and its standard deviation of 16.0%. Figure 1 demonstrates the cVaR dynamics for different distributions of return. Actually, for 5% probability cVaR is around 25% of the portfolio for all distributions, however at lower probabilities a substantial difference emerges — the Student's t-distribution provides the highest estimations, whereas with normal and logistic distributions we have achieved 2-3 times lower values.

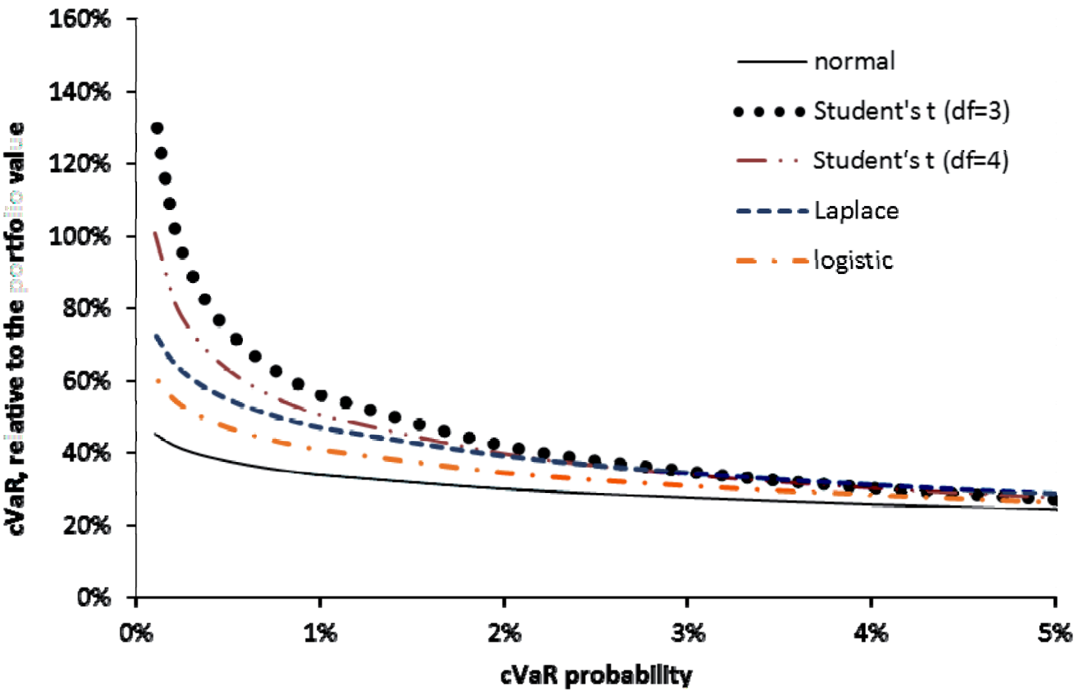

Figure 1. cVaR estimates for the S&P 500 index in 1951-2015

We also can note a definite non-linear dynamics of cVaR, which grows exponentially as the probability decreases. However, if we consider the dependence of the cVaR estimates on the parameters of the distribution (as opposed to the probability), it's actually linear — for a 1% increase (decrease) in the expected return the cVaR estimate drops (grows) by 1%. Also, for a 1% change in the standard deviation the cVaR estimate changes by χ_α %.

Having studied the cVaR dynamics for stocks portfolio, let's also investigate it for a diversified bonds portfolio, for which we can use the exchange-traded fund AGG that tracks the Bloomberg Barclays U.S. Aggregate Bond Index. Using a sample of rolling 12-month returns from September 2004 to September 2016 (145 observations), we can calculate the average sample return of 4.4% and its standard deviation of 2.9%. Figure 2 demonstrated the cVaR dynamics for different distributions of return.

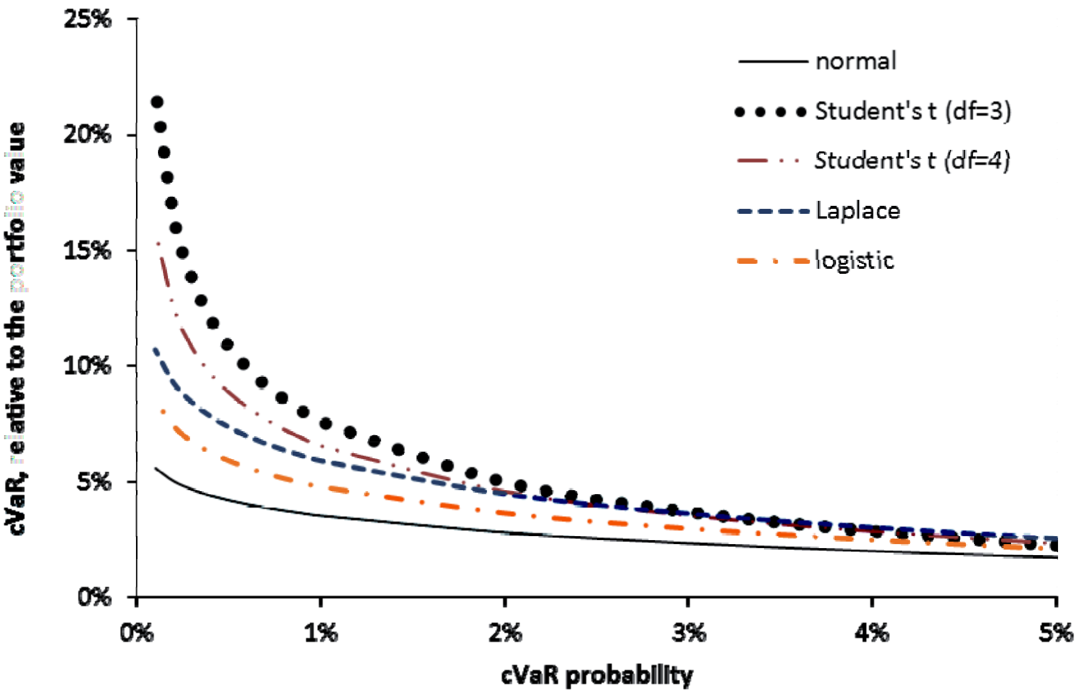

Figure 2. cVaR estimates for the Aggregate Bond index in 2004-2016

Comparing Figure 1 and Figure 2, we can conclude that the shape of each curve is basically the same; it's only scaled down by the factor of 6, which roughly corresponds to the difference in the standard deviation between the S&P 500 and the Bloomberg Barclays U.S. Aggregate Bond Index.

Finally, the generic cVaR formula (8) allows developing a mathematical model for portfolio optimization. The expected value and its standard deviation for any portfolio can we written using the weights of its assets, their expected returns and the covariance matrix. Therefore, we can rewrite (8) as the objective function for portfolio optimization as follows:

$$\min_{\mathbf{w}} cVaR_\alpha = \max_{\mathbf{w}} c_\alpha = \max_{\mathbf{w}} \left\{ \mathbf{w}'\boldsymbol{\mu} - \chi_\alpha \sqrt{\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}} \right\} \quad (9)$$

subject to constraints

$$\mathbf{w}'\mathbf{1} = \sum_{i=1}^{n} w_i = 1, \quad (10)$$

$$w_i^{\min} \le w_i \le w_i^{\max}, i = \overline{1,n}, \quad (11)$$

where $\mathbf{w} = \{w_1, ..., w_n\}$ is the vector of asset weights, $\mathbf{w}'$ is the transposed vector, $\boldsymbol{\mu}$ is the vector of expected returns, $\boldsymbol{\Sigma}$ is the covariance matrix, $w_i^{\min}, w_i^{\max}$ are the lower and the upped bounds on *i*-th asset weight in the portfolio.

(9)–(11) is a non-linear optimization problem, however Khokhlov (2016) suggested a quadratic programming approach that can effectively find all the solutions to (9)–(11). Thus, a quadratic programming algorithm can be used for the portfolio cVaR optimization.

Variance-covariance and historical cVaR

In order to compare the results of variance-covariance cVaR estimation using formulas (4)–(7) with the historical cVaR values calculated with (3), we used a sample of daily returns (756 observations) for the Dow Jones components in 2013–2015 (data source: Yahoo! Finance). Table 1 shows the sample statistics, as well as the historical and estimated cVaR values. We used the normal, the Student's t-distributions with 3 and 4 degrees of freedom and the Laplace distributions to estimate the cVaR. The root mean squared error (RMSE) was used to assess the estimation quality, and the relative RMSE was between 6% and 10%. It appears that the Student's t-distribution provides the cVaR estimates that are closer to the historical cVaR.

Conclusion

Conditional value-at-risk (cVaR) was considered a theoretically appealing coherent risk measure with limited practical implications because of the computational complexity. In this paper we showed that for the elliptical distributions there are quite simple and easy-to-use cVaR formulas, and derived closed-form formulas for some of the most popular distributions. All of these formulas are based on functions that are available in Microsoft Excel™, which enables practitioners to start using cVaR in their routine activities.

The relative RMSE of cVaR @ 5% estimation for Dow Jones components, based on the daily returns sample, is 6–10%, which provides quite a solid base for using cVaR estimated in portfolio management and optimization. For example, the relative RMSE for VaR @ 5% estimation for the same assets is 8–12%. The most accurate estimation is derived using the Student's t-distribution with 3 degrees of freedom.

Based on the results of our research, it is possible to build a more general cVaR portfolio optimization model than the model developed by Rockafellar and Uryasev (2002). We have developed a non-linear cVaR optimization model that can be reduced to a form suitable for quadratic programming algorithms.

Table 1

cVaR @ 5% per $1 estimation for selected stocks in 2013–2015

Ticker	Return Sample Mean	St.dev	Historical cVaR	Estimated cVaR by distribution normal	t (ν = 3)	t (ν = 4)	Laplace
MMM	0.0786%	1.0021%	0.023514	0.019883	0.021628	0.021908	0.022615
IBM	-0.0265%	1.2005%	0.029528	0.025027	0.027117	0.027453	0.028299
GS	0.0593%	1.2908%	0.028652	0.026033	0.028280	0.028641	0.029551
UNH	0.1183%	1.3729%	0.030085	0.027136	0.029526	0.029910	0.030878
HD	0.1152%	1.1409%	0.023297	0.022381	0.024367	0.024686	0.025491
BA	0.1038%	1.3085%	0.028374	0.025952	0.028230	0.028596	0.029519
MCD	0.0563%	0.9423%	0.018148	0.018873	0.020514	0.020777	0.021442
JNJ	0.0662%	0.9189%	0.020706	0.018291	0.019891	0.020148	0.020796
TRV	0.0735%	0.9468%	0.021651	0.018795	0.020443	0.020708	0.021375
CVX	-0.0013%	1.2863%	0.029244	0.026545	0.028784	0.029144	0.030051
UTX	0.0355%	1.0657%	0.024261	0.021627	0.023483	0.023781	0.024532
DIS	0.1113%	1.2297%	0.027446	0.024252	0.026392	0.026736	0.027603
T	0.0282%	0.9495%	0.020982	0.019304	0.020957	0.021223	0.021892
XOM	0.0041%	1.1202%	0.026284	0.023065	0.025016	0.025329	0.026119
PG	0.0375%	0.9289%	0.021009	0.018786	0.020404	0.020663	0.021318
V	0.1069%	1.3580%	0.028501	0.026943	0.029307	0.029687	0.030644
CAT	-0.0165%	1.3448%	0.031889	0.027903	0.030245	0.030621	0.031569
WMT	0.0010%	1.0011%	0.023832	0.020640	0.022383	0.022663	0.023368
DD	0.0783%	1.2910%	0.027506	0.025847	0.028094	0.028455	0.029366
JPM	0.0723%	1.2478%	0.027704	0.025015	0.027187	0.027536	0.028416
AXP	0.0374%	1.2092%	0.027938	0.024567	0.026672	0.027010	0.027863
MRK	0.0542%	1.2109%	0.026819	0.024436	0.026544	0.026883	0.027737
VZ	0.0316%	1.0040%	0.021507	0.020394	0.022141	0.022422	0.023130
NKE	0.1308%	1.3488%	0.023749	0.026513	0.028861	0.029238	0.030189
MSFT	0.1195%	1.5322%	0.032782	0.030410	0.033078	0.033506	0.034587
KO	0.0389%	0.9474%	0.021347	0.019152	0.020801	0.021066	0.021734
PFE	0.0527%	1.1048%	0.024525	0.022261	0.024184	0.024493	0.025272
INTC	0.0911%	1.4159%	0.029813	0.028296	0.030761	0.031157	0.032155
GE	0.0721%	1.1550%	0.022688	0.023103	0.025114	0.025437	0.026251
CSCO	0.0635%	1.3868%	0.029247	0.027971	0.030385	0.030773	0.031750
RMSE, absolute				0.002562	0.001477	0.001509	0.001869
RMSE, relative				9.84%	6.21%	6.40%	7.87%

(Source: author's calculations based on historical prices from Yahoo! Finance)

References:

1. Andreev, A., Kanto, A., Malo, P. (2005). On closed-form calculation of cVaR. *Helsinki School of Economics Working Paper W-389*. <http://epub.lib.aalto.fi/pdf/wp/w389.pdf> (2016, October, 11)
2. Artzner, P., Delbaen, F. Eber, J.-M., Heath, D. (1999). Coherent Measures of Risk. *Mathematical Finance, Vol. 9, No. 3*, 203–228.
3. Jorion, P. (1996). Risk^2: Measuring the Risk in Value at Risk. *Financial Analysts Journal, Vol. 52, No. 6*, 47–56.
4. Khokhlov, V. (2016). Portfolio Value-at-Risk Optimization. *East European Scientific Journal, Vol. 9 (13), Part 2*, 107–113.
5. Linsmeier, T., Pearson, N.D. (2000). Value at Risk. *Financial Analysts Journal, Vol. 56, No. 2*, 47–67.
6. Rockafellar, R.T., Uryasev, S. (2000). Optimization of conditional value-at-risk. *Journal of Risk, Vol. 2*, 21-41.
7. Rockafellar, R.T., Uryasev, S. (2002). Conditional value-at-risk for general loss distributions. *Journal of Banking & Finance, Vol. 26, No. 7*, 1443–1471.
8. Stambaugh, F. (1996). Risk and Value at Risk. *European Management Journal, Vol. 14, No. 6*, 612–621.